

# ENDO-TRIVIAL MODULES FOR FINITE GROUPS WITH DIHEDRAL SYLOW 2-SUBGROUP

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**ABSTRACT.** Let  $k$  be an algebraically closed field of characteristic  $p > 0$  and  $G$  a finite group. We provide a description of the torsion subgroup  $TT(G)$  of the finitely generated abelian group  $T(G)$  of endo-trivial  $kG$ -modules when  $p = 2$  and  $G$  has a dihedral Sylow 2-subgroup  $P$ . We prove that, in the case  $|P| \geq 8$ ,  $TT(G) \cong X(G)$  the group of one-dimensional  $kG$ -modules, except possibly when  $G/O_{2'}(G) \cong \mathfrak{A}_6$ , the alternating group of degree 6; in which case  $G$  may have 9-dimensional simple torsion endo-trivial modules. We also prove a similar result in the case  $|P| = 4$ , although the situation is more involved. Our results complement the tame-representation type investigation of endo-trivial modules started by Carlson-Mazza-Thévenaz in the cases of semi-dihedral and generalized quaternion Sylow 2-subgroups. Furthermore we provide a general reduction result, valid at any prime  $p$ , to recover the structure of  $TT(G)$  from the structure of  $TT(G/H)$ , where  $H$  is a normal  $p'$ -subgroup of  $G$ .

## 1. INTRODUCTION

Let  $G$  be a finite group and  $k$  a field of prime characteristic  $p$  dividing the order of  $G$ . A finitely generated  $kG$ -module  $V$  is called *endo-trivial* if, as  $kG$ -modules,

$$\mathrm{End}_k(V) \cong k_G \oplus Q,$$

where  $k_G$  is the trivial  $kG$ -module and  $Q$  is a projective  $kG$ -module. The tensor product over  $k$  induces a group structure on the set of isomorphism classes of indecomposable endo-trivial  $kG$ -modules, called the group of endo-trivial modules and denoted by  $T(G)$ . This group is finitely generated and it is of particular interest in modular representation theory as it forms an important part of the Picard group of self-equivalences of the stable category of finitely generated  $kG$ -modules. In particular the self-equivalences of Morita type are induced by tensoring with endo-trivial modules.

As a matter of fact, endo-trivial modules have seen a considerable interest since defined by Dade in 1978 [15] as a by-product of the Dade-Glauberman-Nagao correspondence (see [31, §5.12]). In [15] a classification of the endo-trivial modules over finite abelian  $p$ -groups is established. Since then a full classification has been obtained over finite  $p$ -groups through the joint efforts of several authors, see e.g. the survey article [38] and the references therein. Moreover many contributions towards a general classification of endo-trivial modules have been obtained over the past ten years for several families of finite groups (see e.g. [7, 29, 8, 9, 6, 32, 10, 22, 13, 27] and the references therein). However, the problem of describing the structure of  $T(G)$  and its elements for an arbitrary finite

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group  $G$  remains open in general. In particular the problem of determining the structure of the torsion subgroup of  $T(G)$ , denoted by  $TT(G)$ , is a resisting part of the problem.

Provided that a Sylow  $p$ -subgroup  $P$  of  $G$  is neither cyclic, nor dihedral, nor semi-dihedral, the group  $TT(G)$  coincides with the group

$$K(G) = \ker \left( \text{Res}_P^G : T(G) \longrightarrow T(P) : [V] \mapsto [V \downarrow_P^G] \right).$$

(See Lemma 3.3.) In particular, the group  $K(G)$  consists of the classes of the trivial source endo-trivial modules.

Our first main result holds at any prime characteristic  $p$  and relates the structure of the group  $K(G)$  to the structure of  $K(G/H)$ , where  $H$  is a normal  $p'$ -subgroup of  $G$ .

**Theorem 1.1.** *Let  $G$  be a finite group. Assume that the  $p$ -rank of  $G$  is at least 2 and that  $G$  has no strongly  $p$ -embedded subgroups. Let  $H \triangleleft G$  with  $p \nmid |H|$ , and set  $\overline{G} := G/H$ . If  $H^2(\overline{G}, k^\times) = 1$ , then*

$$K(G) = X(G) + \text{Inf}_{\overline{G}}^G(K(\overline{G})) \cong X(G) + K(\overline{G}).$$

However, the main objective of this article is the determination of the structure of the group  $T(G)$ , when the Sylow 2-subgroups of  $G$  are dihedral 2-groups. An investigation of endo-trivial modules in finite- and tame-representation types was started by Mazza-Thévenaz [29] in the cyclic case, Carlson-Mazza-Thévenaz [10] in the generalized quaternion and semi-dihedral cases, and continued by the authors [22] in the Klein-four case. Therefore the dihedral case was the last untreated tame-representation type case. However, we emphasize that [10] does not provide a description of the structure of the group  $K(G)$  in the semi-dihedral case, and which is still an open question.

Since the group  $T(G)$  is finitely generated (see [7, Corollary 2.5]), we may write  $T(G) = TT(G) \oplus TF(G)$ , where  $TF(G)$  denotes a free abelian complement of  $TT(G)$  in  $T(G)$ . We recall that, when a Sylow 2-subgroup of  $G$  is dihedral of order of at least 8, the  $\mathbb{Z}$ -rank, as well as generators for the torsion-free part of  $T(G)$  are known since 1980's. More precisely, since the 2-rank of  $G$  is 2, by [7, Theorem 3.1], the  $\mathbb{Z}$ -rank of  $TF(G)$  coincides with the number of  $G$ -conjugacy classes of Klein-four subgroups of  $G$ , and this number is 2 by [17, Proposition 1.48(iv)]. Then by [1, §4], we have

$$TF(G) \cong \langle [\Omega^1(k_G)], [M] \rangle \cong \mathbb{Z}^2$$

where  $\Omega^1(k_G)$  denotes the first syzygy module of the trivial  $kG$ -module  $k_G$ , and  $M$  is an indecomposable direct summand of the heart of the projective cover of  $k_G$ . (Note that there are two such direct summands and  $M$  can be chosen to be any of them.)

As a consequence, in this article we focus our attention on the determination of the torsion subgroup  $TT(G)$  of  $T(G)$ . We recall that the group  $X(G)$  of one-dimensional  $kG$ -modules endowed with the tensor product  $\otimes_k$  always identifies with a subgroup of  $TT(G)$ . Our main result about the structure of  $TT(G)$  in the dihedral case is the following.

**Theorem 1.2.** *Assume that  $G$  is a finite group with a dihedral Sylow 2-subgroup of order at least 8, and let  $T(G)$  be the abelian group of endo-trivial  $kG$ -modules over an algebraically closed field  $k$  of characteristic 2. Set  $\overline{G} := G/O_2(G)$ . Then the following hold:*

- (a) *If  $\overline{G} \not\cong \mathfrak{A}_6$ , then  $TT(G) = X(G)$ .*
- (b) *If  $\overline{G} \cong \mathfrak{A}_6$ , then either*
  - (i)  *$TT(G) = X(G)$ , or*
  - (ii) *if there exists an indecomposable endo-trivial  $kG$ -module  $V$  such that  $[V] \in TT(G) \setminus X(G)$ , then  $\dim_k V = 9$ ,  $V$  is simple, and  $TT(G)/X(G)$  is an elementary abelian 3-group.*

**Remark 1.3.** Case (i) of Theorem 1.2(b) happens for example for  $G = \mathfrak{A}_6$  (see [8, Theorem 1.2]), whereas Case (ii) happens for example for  $G = 3.\mathfrak{A}_6$ , the triple cover of  $\mathfrak{A}_6$  (see Lemma 6.3). Moreover the central product  $C_9 * 3.\mathfrak{A}_6$  provides an example where there exist classes  $[V] \in TT(G)$  such that  $[V^{\otimes 3}] \in X(G) \setminus \{[k_G]\}$  (see Example 7.2).

Furthermore, in the situation of Theorem 1.2(b)(ii), any simple torsion endo-trivial  $kG$ -module  $V$  of dimension 9 originates from the triple cover  $3.\mathfrak{A}_6$  of the alternating group  $\mathfrak{A}_6$  of degree 6 in the following way. By [33, Theorem],  $E := E(G/\ker(V))$  (the central product of all components of  $G/\ker(V)$ , see [36, Definition 6.6.8]) is quasi-simple and  $V \downarrow_E^{G/\ker(V)}$  remains simple endo-trivial. Therefore, we must have  $E = 3.\mathfrak{A}_6$  since  $3.\mathfrak{A}_6$  is the unique quasi-simple group with a 9-dimensional simple endo-trivial module in characteristic two by [25, Proposition 3.8 and §4].

**Corollary 1.4.** *If  $G$  is a finite group with a dihedral Sylow 2-subgroup of order at least 8 and  $k$  is an algebraically closed field of characteristic 2, then any indecomposable torsion endo-trivial  $kG$ -module is simple, and hence lifts uniquely to an ordinary irreducible character of  $G$ .*

*Proof.* This follows immediately from Theorem 1.2, the fact that  $TT(G)$  consists only of classes of trivial source modules (see Lemma 3.3), and the fact that trivial source modules lift uniquely (see [31, Theorem 4.8.9(iii)]).  $\square$

Our new method, developed to treat the dihedral case of order at least 8, also allows us to finish off the classification of torsion endo-trivial modules for finite groups with Klein-four Sylow 2-subgroups, which we started in [22]. We note that our results in this case are explicit, whereas those recently obtained by Carlson and Thévenaz in [13] (where they treat the general question of computing the group  $K(G)$  for finite groups  $G$  with abelian Sylow  $p$ -subgroups) only provides an algorithmic method to identify the group  $K(G)$ .

**Theorem 1.5.** *Assume that  $G$  is a finite group with a Klein-four Sylow 2-subgroup  $P$ . Further, set  $TT_0(G) := \{[V] \in TT(G) \mid V \text{ indecomposable and } V \in B_0(G)\}$  where  $B_0(G)$  is the principal block of  $G$  and  $\overline{G} := G/O_{2'}(G)$ . Then  $TT_0(G) \cong \mathbb{Z}/3\mathbb{Z}$ , any indecomposable  $kG$ -module  $V$  with  $[V] \in TT(G)$  lifts uniquely to an  $\mathcal{O}G$ -lattice  $\widehat{V}$  affording an ordinary irreducible character  $\chi_{\widehat{V}} \in \text{Irr}(G)$ , and the structure of  $TT(G)$  is as follows:*

- (a) *If  $\overline{G} \cong P$ , then  $TT(G) = X(G)$ .*
- (b) *If  $\overline{G} \cong \mathfrak{A}_4$ , then  $TT(G) = X(G)$ .*
- (c) *If  $\overline{G} \cong \mathfrak{A}_5$ , then*

$$TT(G) \cong TT(G_0) = X(G_0),$$

*where  $G_0$  is a strongly 2-embedded subgroup in  $G$  with  $G_0/O_{2'}(G_0) \cong \mathfrak{A}_4$ . Furthermore, if  $V$  is a non-trivial indecomposable endo-trivial  $kG$ -module such that  $[V] \in TT_0(G)$ , then  $\dim_k(V) = 5$ .*

- (d) *If  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q > 5$  is a power of an odd prime such that  $q \equiv \pm 3 \pmod{8}$  and  $f$  is an odd integer, then*

$$TT(G) \cong X(G) \oplus TT_0(G) \cong X(G) \oplus \mathbb{Z}/3\mathbb{Z}.$$

*Furthermore, if  $V$  is a non-trivial indecomposable endo-trivial  $kG$ -module such that  $[V] \in TT_0(G)$ , then  $\dim_k(V) = (q-1)/2$  when  $q \equiv 3 \pmod{8}$  and  $\dim_k(V) = q$  when  $q \equiv 5 \pmod{8}$ .*

The first main tool used in our investigation is Gorenstein-Walter's classification of finite groups  $G$  with dihedral Sylow 2-subgroups modulo  $O_{2'}(G)$ . Moreover our methods heavily rely on a Theorem of Schur's [31, Theorem 3.5.8] combined with two results of Navarro-Robinson [32], the first of which states that if an endo-trivial module is induced from a proper subgroup, then this subgroup must be strongly  $p$ -embedded in  $G$ , and the second of which states that simple endo-trivial modules over  $p$ -nilpotent groups of  $p$ -rank at least 2 have dimension one. This enables us to reduce our computation of  $TT(G)$  to that of  $TT(G/O_{2'}(G))$  using Theorem 1.1. Finally, we note that our methods require to decompose torsion endo-trivial modules as tensor products of modules over non-proper twisted group algebras, although the final statements of Theorem 1.2 and Theorem 1.5 do not reflect this fact.

The paper is organized as follows. In §2 we introduce the notation, and in §3 preliminary known results on endo-trivial modules, which we will rely on. In §4 we prove Theorem 1.1. In §5 we give general results on finite groups with dihedral Sylow 2-subgroups of order at least 8 and their endo-trivial modules, and §§6–7 are devoted to the proof of Theorem 1.2. Finally in §8 we prove Theorem 1.5.

## 2. NOTATION

Throughout, unless otherwise specified we use the following notation and conventions. We let  $p$  denote a prime number and  $G$  a finite group of order divisible by  $p$ . We assume that  $(K, \mathcal{O}, k)$  is a splitting  $p$ -modular system for all subgroups of  $G$ , that is,  $\mathcal{O}$  is a complete discrete valuation ring of rank one such that its quotient field  $K$  is of characteristic zero, its residue field  $k := \mathcal{O}/\text{rad}(\mathcal{O})$  is of characteristic  $p$ , and both  $K$  and  $k$  are splitting fields for all subgroups of  $G$ . Modules are finitely generated left modules. By an  $\mathcal{O}G$ -lattice, we mean an  $\mathcal{O}G$ -module which is  $\mathcal{O}$ -free of finite rank. For a ring  $R$ , we denote by  $R^\times$  the group of units of  $R$ . We write  $\text{Syl}_p(G)$  for the set of all Sylow  $p$ -subgroups of  $G$ . For a  $p$ -subgroup  $Q$  of  $G$  and  $H \leq G$  with  $H \geq N_G(Q)$ , we denote by  $f = f_{(G, Q, H)}$  the Green correspondence with respect to  $(G, Q, H)$ , see [31, p.276]. For a positive integer  $n$ , we denote by  $C_n$  the cyclic group of order  $n$ , and by  $\mathfrak{A}_n$  the alternating group of degree  $n$ . We write  $Z(G)$  for the center of  $G$ ,  $[G, G]$  for the commutator subgroup of  $G$ . If  $H$  is a normal subgroup of  $G$ , and  $L$  a subgroup of  $G$ , then we write  $G = H \rtimes L$  if  $G$  is a semi-direct product of  $H$  by  $L$ . For two  $kG$ -modules  $M$  and  $M'$ ,  $M \otimes_k M'$  is the tensor product over  $k$ ,  $M^{\otimes n}$  is the tensor product  $M \otimes_k \cdots \otimes_k M$  of  $n$  copies of  $M$ , we write  $M^*$  for the  $k$ -dual of  $M$ , that is  $M^* := \text{Hom}_{kG}(M, k)$ , and we write  $M' | M$  when  $M'$  is (isomorphic to) a direct summand of  $M$ . We denote by  $k_G$  the trivial  $kG$ -module. We write  $H \leq G$  if  $H$  is a subgroup of  $G$ . In such a case, for a  $kG$ -module  $M$  and a  $kH$ -module  $L$ , we denote by  $M \downarrow_H$  and  $L \uparrow^G$ , respectively, the restriction of  $M$  to  $H$  and the induction of  $L$  to  $G$ .

We denote the Schur multiplier of  $G$  by  $M(G) := H^2(G, \mathbb{C}^\times)$ . For a 2-cocycle  $\alpha \in Z^2(G, k^\times)$ , we denote by  $[\alpha] \in H^2(G, k^\times)$  the cohomology class of  $\alpha$ , and by  $k^\alpha G$  the twisted group algebra of  $G$  over  $k$  with respect to  $\alpha$ . Then for a  $k^\alpha G$ -module  $M$  and  $H \leq G$  we write  $M \downarrow_H$  or  $M \downarrow_{k^\alpha H}^{k^\alpha G}$  for the restriction of  $M$  from  $G$  to  $H$ . Assume that  $N \triangleleft G$ . For a 2-cocycle  $\bar{\alpha} \in Z^2(G/N, k^\times)$  we denote by  $\text{Inf}_{G/N}^G(\bar{\alpha}) \in Z^2(G, k^\times)$  the inflation of  $\bar{\alpha}$  from  $G/N$  to  $G$ , and by  $\text{Inf}_{k^\alpha(G/N)}^{k^\alpha G}(M)$ , the inflation of a  $k^{\bar{\alpha}}(G/N)$  module  $M$  to a  $k^\alpha G$ -module with  $\alpha = \text{Inf}_{G/N}^G(\bar{\alpha})$ . For a  $k(G/N)$ -module  $M$  we write simply  $\text{Inf}_{G/N}^G(M)$  for the inflation of  $M$  from  $G/N$  to  $G$ . We denote by  $B_0(G)$ , the principal block of  $G$ , and by  $\text{Irr}(G)$  the set of all irreducible ordinary characters of  $G$ . For a  $p$ -block  $B$  of  $G$ ,

we also denote by  $\text{Irr}(B)$  the set of all characters in  $\text{Irr}(G)$  which belong to  $B$ . We write  $1_G$  for the trivial ordinary or Brauer character of  $G$ .

We say that a  $kG$ -module  $M$  is a *trivial source* module if it is a direct sum of indecomposable  $kG$ -modules, all of whose sources are trivial modules, see [37, p.218]. It is known that a trivial source  $kG$ -module  $M$  lifts uniquely to a trivial source  $\mathcal{O}G$ -lattice, which we denote by  $\widehat{M}$ , see [31, Theorem 4.8.9(iii)]. Then, we denote by  $\chi_{\widehat{M}}$  the ordinary character of  $G$  afforded by  $\widehat{M}$ . For a non-negative integer  $m$  and a positive integer  $n$ , we write  $n_p = p^m$  if  $p^m | n$  and  $p^{m+1} \nmid n$ .

For further standard notation and terminology, we refer the reader to the books [31, 37].

### 3. PRELIMINARY RESULTS

**3.1. Endo-trivial modules.** A  $kG$ -module  $V$  is called *endo-trivial* provided

$$\text{End}_k(V) \cong V^* \otimes_k V \cong k_G \oplus (\text{proj})$$

as  $kG$ -modules where  $(\text{proj})$  denotes a projective direct summand (possibly the zero module).

Any endo-trivial  $kG$ -module  $V$  splits as the direct sum  $V = V_0 \oplus (\text{proj})$  where  $V_0 := \Omega^0(V)$ , the projective-free part of  $V$ , is indecomposable and endo-trivial. The relation  $U \sim V \Leftrightarrow U_0 \cong V_0$  is an equivalence relation on the class of endo-trivial  $kG$ -modules, and we let  $T(G)$  denote the resulting set of equivalence classes (which we denote by square brackets). Then  $T(G)$ , endowed with the law  $[U] + [V] := [U \otimes_k V]$ , is an abelian group called the *group of endo-trivial modules of  $G$* . The zero element is the class  $[k_G]$  and  $-[V] = [V^*]$ .

Notice that if  $p \nmid |G|$ , then any  $kG$ -module is endo-trivial, but the above construction of the group  $T(G)$  is not valid any more.

The group  $T(G)$  is known to be a finitely generated abelian group, see e.g. [7, Corollary 2.5]. Therefore, we may write  $T(G) = TT(G) \oplus TF(G)$ , where  $TT(G)$  is the torsion subgroup of  $T(G)$  (hence a finite group) and  $TF(G)$  is a torsion-free complement.

We let  $X(G)$  denote the group of one-dimensional  $kG$ -modules endowed with the tensor product  $\otimes_k$ , and recall that  $X(G) \cong (G/[G, G])_{p'}$ . Then by identifying a one-dimensional module with its class in  $T(G)$ , we consider  $X(G)$  as a subgroup of  $T(G)$ .

Furthermore, define  $T_0(G) := \{[V] \in T(G) \mid V \text{ indecomposable and } V \in B_0(G)\}$ . Then  $T_0(G) \leq T(G)$  by [11, Proposition 9.1]. Denote by  $TT_0(G)$  the torsion subgroup of  $T_0(G)$ , and by  $X_0(G)$  the set of one-dimensional  $kG$ -modules belonging to  $B_0(G)$ . Clearly  $X_0(G) \leq TT_0(G) \leq TT(G)$ .

**Remark 3.1.** Because the dimension of projective  $kG$ -modules is divisible by  $|G|_p$ , if  $M$  is an endo-trivial  $kG$ -module, then  $\dim_k(V) \equiv \pm 1 \pmod{|G|_p}$  if  $p$  is odd; and  $\dim_k(V) \equiv \pm 1 \pmod{\frac{1}{2}|G|_2}$  if  $p = 2$ . Moreover if  $V$  is indecomposable with trivial source, then  $\dim_k(V) \equiv 1 \pmod{|G|_p}$ . In particular, indecomposable endo-trivial  $kG$ -modules have the Sylow  $p$ -subgroups of  $G$  as their vertices, and hence lie in  $p$ -blocks of full defect.

**Lemma 3.2.** *Let  $H$  be a subgroup of  $G$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ .*

- (a) *If  $V$  is an endo-trivial  $kG$ -module, then  $V \downarrow_H$  is endo-trivial. Moreover, if  $H \geq P$ , then  $V$  is endo-trivial if and only if  $V \downarrow_H$  is endo-trivial.*
- (b) *If  $p \mid |H|$ , then restriction induces a group homomorphism*

$$\text{Res}_H^G : T(G) \longrightarrow T(H) : [V] \mapsto [V \downarrow_H].$$



If, moreover,  $H \geq N_G(P)$ , then  $\text{Res}_H^G$  is injective, and for any  $[V] \in T(G)$  with  $V$  indecomposable,  $\text{Res}_H^G([V]) = [f_H(V)]$ , where  $f_H := f_{(G,P,H)}$ .

- (c) If  $H \triangleleft G$  such that  $p \nmid |H|$  and  $V$  is a  $k(G/H)$ -module, then  $\text{Inf}_{G/H}^G(V)$  is endo-trivial if and only if  $V$  is endo-trivial. Moreover the inflation from  $G/H$  to  $G$  induces an injective group homomorphism

$$\text{Inf}_{G/H}^G : T(G/H) \longrightarrow T(G) : [V] \mapsto [\text{Inf}_{G/H}^G(V)].$$

In particular, we may consider  $TT(G/H) \leq TT(G)$ .

*Proof.* Parts (a) and (b) are given by [7, Proposition 2.6], and part (c) is given by [26, Lemma 3.2(1)].  $\square$

Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . We fix the notation

$$K(G) := \ker(\text{Res}_P^G : T(G) \longrightarrow T(P)).$$

In fact, in most cases, the torsion subgroup of  $T(G)$  is equal to  $K(G)$ , and has the following characterizations, which we will use throughout.

**Lemma 3.3.** *Let  $P$  be a Sylow  $p$ -subgroup of  $G$ .*

- (a) *If for any  $x \in G$ ,  $P \cap x^{-1}Px$  is non-trivial, then  $K(G) = X(G)$ . In particular  $K(N_G(P)) = X(N_G(P))$ .*
- (b) *The group  $K(G)$  is exactly the set of classes of indecomposable trivial source endo-trivial  $kG$ -modules, and*

$$K(G) = \{ [V] \in T(G) \mid \exists \text{ a module } M \in X(N_G(P)) \text{ with } V_0 = f^{-1}(M) \},$$

where  $f := f_{(G,P,N_G(P))}$ . In particular,  $K(G) \leq TT(G)$  and we may consider  $K(G)$  as a subgroup of  $K(N_G(P)) = X(N_G(P))$  via the injective homomorphism  $\text{Res}_{N_G(P)}^G$ .

- (c) *Furthermore, provided  $P$  is neither cyclic, nor semi-dihedral, nor generalized quaternion, then  $K(G) = TT(G)$ .*

*Proof.* (a) This follows from [29, Lemma 2.6].

(b) By definition  $K(G)$  consists of the classes of indecomposable trivial source endo-trivial  $kG$ -modules. Hence the first claim is straightforward from (a) together with Lemma 3.2(b). Next the number of isomorphism classes of indecomposable trivial source  $kG$ -modules with vertex  $P$  is finite, hence  $K(G)$  is a finite group, so that we must have  $K(G) \leq TT(G)$ . The last claim follows from Lemma 3.2(b).

(c) The claim is given by [9, Lemma 2.3].  $\square$

Our objective in this article is to consider groups with dihedral Sylow 2-subgroups only, therefore in this case Lemma 3.3(c) allows us to identify  $TT(G)$  with  $K(G)$ .

Finally in order to detect whether a trivial source module is endo-trivial, we have the following character-theoretic criterion.

**Theorem 3.4** ([24, Theorem 2.2]). *Let  $V$  be an indecomposable trivial source  $kG$ -module. Then  $V$  is endo-trivial if and only if  $\chi_{\widehat{V}}(u) = 1$  for any non-trivial  $p$ -element  $u \in G$ .*

**3.2. Strongly  $p$ -embedded subgroups.** Recall that a subgroup  $H$  of  $G$  is said to be *strongly  $p$ -embedded in  $G$*  if  $H \not\leq G$ ,  $p \mid |H|$  and  $p \nmid |H \cap x^{-1}Hx|$  for any  $x \in G \setminus H$ . Note that any strongly  $p$ -embedded subgroup of  $G$  contains the normalizer in  $G$  of a Sylow  $p$ -subgroup. Moreover the operations of induction and restriction induce equivalences of the stable module categories  $\mathbf{stmod}(kH)$  and  $\mathbf{stmod}(kG)$  if  $H$  is a strongly  $p$ -embedded subgroup of  $G$ .

**Lemma 3.5** ([29, Lemma 2.7(2)]). *Let  $H$  be a strongly  $p$ -embedded subgroup of  $G$ , and let  $P$  be a Sylow  $p$ -subgroup of  $H$ . Then  $\text{Res}_H^G : T(G) \rightarrow T(H)$  is an isomorphism. Moreover, the inverse map is induced by induction, and, more precisely, on the indecomposable endo-trivial modules by the Green correspondence  $f_H := f_{(G,P,H)}$ , that is*

$$\begin{aligned} T(G) &= \{ [W \uparrow^G] \mid [W] \in T(H) \} \\ &= \{ [f_H^{-1}(W)] \mid W \text{ is an indecomposable endo-trivial } kH\text{-module} \}. \end{aligned}$$

In particular  $K(G) \cong K(H)$ .

The following result due to G. Navarro and G.R. Robinson is essential for our purpose because the structure of finite groups which have strongly  $p$ -embedded subgroups is in a sense very restricted.

**Lemma 3.6.** *Let  $H$  be a proper subgroup of  $G$ , and assume that  $V$  and  $W$  are  $kG$ - and  $kH$ -modules, respectively, with  $V = W \uparrow^G$ . Then the following are equivalent:*

- (1)  $V$  is endo-trivial.
- (2)  $W$  is endo-trivial and  $H$  is a strongly  $p$ -embedded subgroup of  $G$ .

*Proof.* The implication (1) implies (2) is given by [29, Lemma 1(iv)]. The converse is straightforward from Lemma 3.5.  $\square$

#### 4. RECOVERING $K(G)$ VIA INFLATION FROM A NORMAL $p'$ -SUBGROUP

Before starting our investigation of endo-trivial modules over finite groups with dihedral Sylow 2-subgroups, we develop a general method enabling us to recover the subgroup  $K(G)$  of  $T(G)$  using inflation from a quotient by a normal  $p'$ -subgroup of  $G$ .

In order to set up the technical notation for this section, we start by recalling and slightly generalizing well-known results of Schur.

**Lemma 4.1** (Schur). *Let  $F$  be an algebraically closed field of arbitrary characteristic, let  $H \triangleleft G$  and set  $\overline{G} := G/H$ . Let  $Y$  be an  $n$ -dimensional simple  $FH$ -module which is  $G$ -invariant. Then the following hold:*

- (a)  $Y$  extends to an  $F^\alpha G$ -module  $\widehat{Y}$ , where  $\alpha \in Z^2(G, k^\times)$ , which satisfies the following two conditions: For any  $h \in H$ , any  $g \in G$  and any  $y \in \widehat{Y}$ ,

- (i)  $(hg) \cdot y = h \cdot (g \cdot y)$ , and
- (ii)  $(gh) \cdot y = g \cdot (h \cdot y)$ .

Moreover,  $\alpha(hg, h'g') = \alpha(g, g')$  for all  $h, h' \in H$  and all  $g, g' \in G$ , so that  $\alpha$  defines a 2-cocycle  $\overline{\alpha} : \overline{G} \times \overline{G} \rightarrow k^\times : (gH, g'H) \mapsto \alpha(g, g')$ , i.e.  $\alpha = \text{Inf}_{\overline{G}}^G(\overline{\alpha})$ , and we have  $[\overline{\alpha}^{n|H|}] = 1 \in H^2(\overline{G}, k^\times)$ .

- (b) Assume that  $\widehat{Y}$  is an  $F^\alpha G$ -module extending  $Y$ , defined by a 2-cocycle  $\alpha \in Z^2(G, k^\times)$  as in (a). If  $X$  is an  $FG$ -module such that

$$X \downarrow_H \cong Y \oplus \cdots \oplus Y,$$

the direct sum of  $e \geq 1$  copies of  $Y$ , then there exists an  $F^{\bar{\alpha}^{-1}}\bar{G}$ -module  $Z$  such that, as  $FG$ -modules,

$$X \cong \hat{Y} \otimes_F \operatorname{Inf}_{F^{\bar{\alpha}^{-1}}\bar{G}}^{F^{\bar{\alpha}^{-1}}G}(Z).$$

*Proof.* Part (a) is exactly Schur's result [31, Theorem 3.5.7]. Part (b) is a generalization of Schur's theorem [31, Theorem 3.5.8(i)]. More specifically, although [31, Theorem 3.5.8(i)] is stated for a module  $X$  which is *simple*, its proof only requires the assumption that  $X \downarrow_H \cong Y \oplus \cdots \oplus Y$  in order to obtain the conclusion that there exists an  $F^{\bar{\alpha}^{-1}}(\bar{G})$ -module  $Z$  such that  $X \cong \hat{Y} \otimes_F \operatorname{Inf}_{F^{\bar{\alpha}^{-1}}\bar{G}}^{F^{\bar{\alpha}^{-1}}G}(Z)$ .  $\square$

**Remark 4.2.**

- (a) We recall that inflation of 2-cocycles  $Z^2(\bar{G}, k^\times) \rightarrow Z^2(G, k^\times) : \alpha \mapsto \operatorname{Inf}_{\bar{G}}^G(\alpha)$  induces an inflation homomorphism  $\operatorname{Inf}_{\bar{G}}^G : H^2(\bar{G}, k^\times) \rightarrow H^2(G, k^\times) : [\bar{\alpha}] \mapsto [\operatorname{Inf}_{\bar{G}}^G(\bar{\alpha})]$  in cohomology, but the latter need not be injective in general. Therefore, it may happen that  $\operatorname{Inf}_{F^{\bar{\alpha}^{-1}}\bar{G}}^{F^{\bar{\alpha}^{-1}}G}(Z)$  is in fact a module over the non-twisted group algebra  $FG$ , while  $Z$  is a module over the twisted group algebra  $F^{\bar{\alpha}^{-1}}\bar{G} \not\cong FG$ .
- (b) In fact, more general statements than Lemma 4.1(b) can be discussed by making use of results of E.C. Dade. We refer the reader to the book of A. Marcus [28, §2.3.B].

**Lemma 4.3.** *Assume that  $G$  has no strongly  $p$ -embedded subgroups and  $V$  is an indecomposable endo-trivial  $kG$ -module. If  $H \triangleleft G$  such that  $p \nmid |H|$  and  $L$  is a composition factor of  $V \downarrow_H$ , then  $L$  is  $G$ -invariant.*

*Proof.* Set  $\tilde{G} := T_G(L)$ , the stabilizer of  $L$  in  $G$ , and let  $B$  be the block of  $kG$  to which  $V$  belongs. Let  $\tilde{B}$  be the block of  $k\tilde{G}$  such that  $\tilde{B}$  is the Fong-Reynolds correspondent of  $B$ , see [31, Theorem 5.5.10]. Then,  $B$  and  $\tilde{B}$  are Morita equivalent. Write  $\tilde{V}$  for the  $k\tilde{G}$ -module in  $\tilde{B}$  corresponding to  $V$  via this Morita equivalence. Then,  $V = \tilde{V} \uparrow^G$ . Thus Lemma 3.6 yields that  $\tilde{V}$  is endo-trivial. If  $\tilde{G} \neq G$ , then it follows from Lemma 3.6 that  $\tilde{G}$  is strongly  $p$ -embedded in  $G$ , which is a contradiction.  $\square$

**Theorem 4.4.** *Assume that the  $p$ -rank of  $G$  is at least 2 and that  $G$  has no strongly  $p$ -embedded subgroups. Let  $H \triangleleft G$  with  $p \nmid |H|$ , and set  $\bar{G} := G/H$ . Let  $V$  be an indecomposable endo-trivial  $kG$ -module. Then the following hold:*

- (a) *There exists a 2-cocycle  $\bar{\alpha} \in Z^2(\bar{G}, k^\times)$  such that*

$$V \cong 1b \otimes_k W,$$

*where  $1b$  is a one-dimensional  $k^\alpha G$ -module for  $\alpha := \operatorname{Inf}_{\bar{G}}^G(\bar{\alpha})$  and  $W := \operatorname{Inf}_{k^{\bar{\alpha}^{-1}}\bar{G}}^{k^{\bar{\alpha}^{-1}}G}(Z)$  for a  $k^{\bar{\alpha}^{-1}}\bar{G}$ -module  $Z$ . Moreover, if  $P \in \operatorname{Syl}_p(G)$  and  $\bar{P} := HP/H = (H \rtimes P)/H$ , then we have  $[\alpha \downarrow_{H \rtimes P}] = 1 \in H^2(H \rtimes P, k^\times)$  and  $[\bar{\alpha} \downarrow_{\bar{P}}] = 1 \in H^2(\bar{P}, k^\times)$ .*

- (b) *Keep the notation of (a), and assume moreover that  $[V] \in K(G)$ . Set  $n := |[\bar{\alpha}]|$ , the order of  $[\bar{\alpha}] \in H^2(\bar{G}, k^\times)$ . Then  $1c := (1b)^{\otimes n}$  is a one-dimensional  $kG$ -module,  $Z^{\otimes n}$  is a trivial source  $k\bar{G}$ -module, and  $W^{\otimes n} = \operatorname{Inf}_{\bar{G}}^G(Z^{\otimes n})$  is a trivial source endo-trivial  $kG$ -module. In other words,*

$$[V^{\otimes n}] = [1c] + [W^{\otimes n}] \in X(G) + \operatorname{Inf}_{\bar{G}}^G(K(\bar{G})).$$

*Proof.* (a) Let  $L$  be a composition factor of  $V \downarrow_H$ . Then, by Lemma 4.3,  $L$  is  $G$ -invariant. Let  $B$  and  $b$  be the blocks of  $kG$  and  $kH$ , respectively, to which  $V$  and  $L$  belong. So



clearly  $B$  covers  $b$ . Denote by  $\theta \in \text{Irr}(b)$  the ordinary irreducible character corresponding to  $L$ , and hence  $\text{Irr}(b) = \{\theta\}$ .

Since  $B$  covers  $b$ ,  $\text{Irr}(b) = \{\theta\}$ ,  $\theta$  is  $G$ -invariant and  $H$  is a  $p'$ -group, it follows from [31, Lemma 5.5.8(ii)] that

$$V \downarrow_H \cong L \oplus \cdots \oplus L.$$

Now, since  $L$  is a  $G$ -invariant simple  $kH$ -module, we know from another Lemma 4.1(a) that there exist a 2-cocycle  $\bar{\alpha} \in Z^2(\bar{G}, k^\times)$  and a  $k^\alpha G$ -module  $\hat{L}$ , for  $\alpha = \text{Inf}_{\bar{G}}^G(\bar{\alpha})$ , such that  $\hat{L} \downarrow_H = L$ . Then by Lemma 4.1(b), there exists a  $k^{\alpha^{-1}} G$ -module  $W$  such that

$$V \cong \hat{L} \otimes_k W,$$

where  $W := \text{Inf}_{k^{\alpha^{-1}} \bar{G}}^{k^{\alpha^{-1}} G}(Z)$  for a  $k^{\alpha^{-1}} \bar{G}$ -module  $Z$ .

Then, we have  $[\bar{\alpha} \downarrow_{\bar{P}}] = 1$  as an element of  $H^2(\bar{P}, k^\times)$  by [31, Proof of Theorem 3.5.11(ii)], and therefore  $[\alpha \downarrow_{H \rtimes P}] = 1$  as an element of  $H^2(H \rtimes P, k^\times)$ . This implies that

$$V \downarrow_{H \rtimes P} \cong \hat{L} \downarrow_{k(H \rtimes P)} \otimes_k W \downarrow_{k(H \rtimes P)},$$

where all three modules are modules over the (genuine non-twisted) group algebra  $k(H \rtimes P)$ . Then, by Lemma 3.2(a),  $V \downarrow_{H \rtimes P}$  is endo-trivial. Hence, both  $\hat{L} \downarrow_{k(H \rtimes P)}$  and  $W \downarrow_{k(H \rtimes P)}$  are endo-trivial by [32, Lemma 1(iii)]. In addition, since  $L$  is simple and  $\hat{L}$  is an extension of  $L$ ,  $\hat{L} \downarrow_{k(H \rtimes P)}$  is simple as well. Thus, as the  $p$ -rank of  $G$  is assumed to be at least 2, [32, Theorem] yields  $\dim L = \dim \hat{L} = 1$ . So, we set  $1b := \hat{L}$ , and (a) follows. (b) First since  $V$  is an indecomposable endo-trivial  $kG$ -module with  $[V] \in K(G)$ , we have  $[V^{\otimes n}] \in K(G)$  as well. Then, since  $|\bar{\alpha}| = n$ , we have by (a) that

$$V^{\otimes n} \cong 1c \otimes_k W^{\otimes n},$$

where, by Lemma 4.1(a),  $1c := (1b)^{\otimes n}$  is a one-dimensional (genuine non-twisted)  $kG$ -module and  $W^{\otimes n} = \text{Inf}_{\bar{G}}^G(Z)$  is a non-twisted  $kG$ -module inflated from the non-twisted  $k\bar{G}$ -module  $Z^{\otimes n}$ . Since  $V^{\otimes n}$  is endo-trivial, again by [32, Lemma 1(iii)], both  $1c$  and  $W^{\otimes n}$  are endo-trivial  $kG$ -modules, and thus  $W^{\otimes n}$  is also endo-trivial as a  $k\bar{G}$ -module by Lemma 3.2(c). Now in  $T(G)$  we have

$$[V^{\otimes n}] = [1c] + [W^{\otimes n}],$$

where  $[1c] \in X(G) \leq K(G)$ . Therefore it remains to prove that  $[W^{\otimes n}] \in \text{Inf}_{\bar{G}}^G(K(\bar{G}))$ . But this is clear. Indeed, since  $[W^{\otimes n}] = [V^{\otimes n}] - [1c] \in K(G)$ ,  $W^{\otimes n}$  must be a trivial source  $kG$ -module (possibly the direct sum of an indecomposable endo-trivial module and projective modules if  $n > 1$ ) by Lemma 3.3(b) and therefore so is the  $k\bar{G}$ -module  $Z^{\otimes n}$ , that is,  $[W^{\otimes n}] = \text{Inf}_{\bar{G}}^G([Z^{\otimes n}]) \in \text{Inf}_{\bar{G}}^G(K(\bar{G}))$ .  $\square$

As a corollary we obtain Theorem 1.1 of the introduction.

*Proof of Theorem 1.1.* Since  $H^2(\bar{G}, k^\times) = 1$ , the integer  $n$  in Theorem 4.4(b) is equal to 1. Hence the claim follows by identifying  $K(\bar{G})$  with  $\text{Inf}_{\bar{G}}^G(K(\bar{G}))$ .  $\square$

## 5. GROUPS WITH DIHEDRAL SYLOW 2-SUBGROUPS

Throughout this section we assume that  $P$  be a dihedral Sylow 2-subgroup of  $G$  of order at least 8. Gorenstein and Walter proved in [18] (see also [16, Theorem on p.462]) that  $\bar{G} := G/O_{2'}(G)$  is isomorphic to either

- (1)  $P$ ,

- (2) the alternating group  $\mathfrak{A}_7$ , or
- (3) a subgroup of  $\text{PTL}(2, q)$  containing  $\text{PSL}(2, q)$ , where  $q$  is a power of an odd prime.

**Hypotheses 5.1.** For the purposes of our computations, we split case (3) above in further subcases and say that  $G$  satisfies the hypothesis:

- (D1) if  $\overline{G} \cong P$ ;
- (D2) if  $\overline{G} \cong \mathfrak{A}_7$ ;
- (D3) if  $\overline{G} \cong \text{PSL}(2, 9) \cong \mathfrak{A}_6$ ;
- (D4) if  $\overline{G} \cong \text{PGL}(2, 9) \cong \mathfrak{A}_{6.2_2}$ ;
- (D5) if  $\overline{G} \cong \text{PSL}(2, q)$ , where  $q$  is a power of an odd prime with  $q \neq 9$ , and  $q \equiv \pm 1 \pmod{8}$ ;
- (D6) if  $\overline{G} \cong \text{PGL}(2, q)$ , where  $q$  is a power of an odd prime with  $q \neq 9$ ;
- (D7) if  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q$  is a power of an odd prime with  $q \neq 9$ ,  $q \equiv \pm 1 \pmod{8}$ , and  $f > 1$  is odd;
- (D8) if  $\overline{G} \cong \text{PGL}(2, q) \rtimes C_f$ , where  $q$  is a power of an odd prime with  $q \neq 9$ , and  $f > 1$  is odd.

The splitting of case (3) into (D3)-(D8) follows from the fact that the structure of  $\text{PTL}(2, q)$ , where  $q = r^m$  is a power of an odd prime  $r$ , is well-known:  $\text{PTL}(2, q) \cong \text{PGL}(2, q) \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_r)$ , where  $\text{Gal}(\mathbb{F}_q/\mathbb{F}_r)$  is cyclic of order  $m$ . Moreover [36, Chapter 6 (8.9)] shows that  $f$  is odd.

**Lemma 5.2.** *There are no strongly 2-embedded subgroups in  $G$ .*

*Proof.* This follows from the Bender-Suzuki Theorem [2, Satz 1] (cf. [35]) and also a result of Gorenstein-Walter [18], see [16, Theorem on p.462], too.  $\square$

**Lemma 5.3.** *Set  $h := |\text{H}^2(\overline{G}, k^\times)|$ . Then*

$$h = \begin{cases} 3 & \text{if } \overline{G} \in \{\mathfrak{A}_6, \mathfrak{A}_7, \text{PGL}(2, 9)\}, \\ 1 & \text{otherwise.} \end{cases}$$

*Proof.* Since  $k$  has characteristic 2, it follows from [41, Proposition 3.2] (see also [21, Lemma 5] and [31, Lemma 3.5.4(ii)]) that  $\text{H}^2(\overline{G}, k^\times) \cong M(\overline{G})_{2'}$ , where  $M(\overline{G})_{2'}$  is the  $2'$ -part of  $M(\overline{G}) = \text{H}^2(\overline{G}, \mathbb{C}^\times)$ .

If (D1) holds, then  $h = 1$  by [31, Theorem 2.7.4]. If (D2) or (D3) holds, then  $h = 3$  by [14, p.10 and p.4, respectively].

Assume (D4) holds. Since  $\overline{G}/\text{PSL}(2, 9)$  is cyclic and  $\text{PSL}(2, 9)$  is perfect, we know by [19, Theorem 3.1] that  $|M(\overline{G})| \mid |M(\text{PSL}(2, 9))| = 6$ . So  $h = 1$  or  $h = 3$ . Then [40,  $A_6 \pmod{2}$ ] yields  $h = 3$ .

Next assume (D5) holds. It is known that if  $q \neq 9$  is a power of an odd prime, then  $|M(\text{PSL}(2, q))| = 2$  by a result of R. Steinberg in [20, Theorem 4.9.1(ii)]. Hence, we have  $h = 1$ .

Next, consider the particular case that  $q = 3$  when (D6) or (D8) holds (note that  $q \neq 3$  if (D7) holds). Assume first that (D6) occurs. Then, in the former case,  $\overline{G} \cong \text{PGL}(2, 3) \cong \mathfrak{S}_4$  and by Schur's result [20, Theorem 4.3.8(i)] we have that  $|M(\mathfrak{S}_4)| = 2$ , so that  $h = 1$ . Assume that (D8) occurs. It follows from [19, Theorem 3.1(i)] that

$$|M(\overline{G})| \mid (|M(\mathfrak{S}_4)| \cdot |M(\mathfrak{S}_4/[\mathfrak{S}_4, \mathfrak{S}_4])|) = 2 \times 2 = 4.$$

Hence we have  $h = 1$ .

Finally assume that **(D6)**, **(D7)** or **(D8)** holds with  $q > 3$ . Then

$$\overline{G}/\mathrm{PSL}(2, q) \cong \begin{cases} C_2 & \text{if } \mathbf{(D6)} \text{ holds,} \\ C_f & \text{if } \mathbf{(D7)} \text{ holds,} \\ C_{2f} & \text{if } \mathbf{(D8)} \text{ holds,} \end{cases}$$

which, in particular, is cyclic in all three cases. Thus, as  $\mathrm{PSL}(2, q)$  is perfect, it follows from [19, Theorem 3.1(i)] that  $|M(\overline{G})| \mid |M(\mathrm{PSL}(2, q))| = 2$ . Therefore we obtain  $h = 1$ .  $\square$

## 6. TORSION ENDO-TRIVIAL MODULES IN THE DIHEDRAL CASE: THE BASIC EXAMPLES

We now turn to the description of  $TT(G)$  for groups  $G$  with dihedral Sylow 2-subgroup of order at least 8. First we investigate the case when  $O_{2'}(G)$  is trivial and prove that torsion endo-trivial modules are always one-dimensional in this case. Throughout this section we use the notations  $G$ ,  $P$  and  $\overline{G}$  as in §5.

**Proposition 6.1.** *If  $O_{2'}(G) = 1$ , then  $TT(G) = K(G) = X(G)$ .*

*Proof.* By assumption, we have  $G = \overline{G}$ , thus we may go through the possibilities for  $G$  according to Hypotheses 5.1.

If **(D1)** holds, i.e.  $G = P$ , then  $TT(G) = \{[k_G]\}$  by [12, Theorem 5.4]. If **(D2)** holds, i.e.  $G = \mathfrak{A}_7$ , then  $TT(G) = \{[k_G]\}$  by [8, Theorem B(a)].

Now assume that  $G$  satisfies one of **(D3)** to **(D8)**. Set  $N := N_G(P)$ . As  $P$  is dihedral of order at least 8, its automorphism group is a 2-group, so that

$$N = PC_G(P) = P \times O_{2'}(C_G(P)).$$

In particular, if  $G$  satisfies **(D3)**, **(D4)**, **(D5)** or **(D6)**, then  $N = P$ , and hence  $X(N) = \{[k_N]\}$ , so that Lemma 3.3(b) yields  $TT(G) = K(G) = \{[k_G]\} = X(G)$ .

Finally assume that  $G$  satisfies **(D7)** or **(D8)**. Then  $N = P \times C_f$ , so that  $X(N) \cong C_f$ . But clearly  $X(G) \cong C_f$  as well, so that the  $kG$ -Green correspondents of the one-dimensional  $kN$ -modules are all one-dimensional. Hence

$$TT(G) = K(G) = X(G) \cong C_f$$

by Lemma 3.3(b).  $\square$

As a consequence, we see that any torsion endo-trivial module of a finite group with dihedral Sylow 2-subgroup of order at least 8 which lies in the principal block has to be one-dimensional.

**Corollary 6.2.** *There is an isomorphism of groups  $TT_0(G) \cong X_0(G)$ .*

*Proof.* Since  $O_{2'}(G)$  acts trivially on the principal block, Lemma 3.2(c) yields

$$TT_0(G) = \mathrm{Inf}_{G/O_{2'}(G)}^G (TT_0(G/O_{2'}(G))) .$$

Now, by Proposition 6.1,  $TT_0(G/O_{2'}(G)) = X_0(G/O_{2'}(G))$ . The claim follows.  $\square$

Next we consider the triple covers of  $\mathfrak{A}_6$ ,  $\mathfrak{A}_7$  and  $\mathrm{PGL}(2, 9)$  whose Schur multipliers have non-trivial  $2'$ -parts as seen in Lemma 5.3. The next lemma also shows that  $TT(G)$  is not isomorphic to  $TT(G/O_{2'}(G))$  via inflation in general.

**Lemma 6.3.** (a) *Let  $H := 3.\mathfrak{A}_6$ , then  $TT(H) = \{[k_H], [9_1], [9_2]\} \cong \mathbb{Z}/3\mathbb{Z}$ , where  $9_1$  and  $9_2$  are mutually dual 9-dimensional, faithful, simple and trivial source  $kH$ -modules.*

(b) *Let  $G := 3.\mathfrak{A}_7$ , then  $TT(G) = TT_0(G) = X(G) = \{[k_G]\}$ .*

(c) Let  $R := 3.\text{PGL}(2, 9)$ , then  $TT(R) = TT_0(R) = X(R) = \{[k_R]\}$ .

In the following proof, an ordinary irreducible character of degree  $d$  of a group  $G$  is denoted by  $\chi_d$ , whereas irreducible Brauer characters are denoted by their degrees, and are identified with the corresponding simple  $kG$ -modules. Moreover ordinary irreducible characters are labelled according to [40, Decomposition Matrices]. We note that the above result about  $3.\mathfrak{A}_6$  and  $3.\mathfrak{A}_7$  appears in [27, Proposition 6.1], where it was obtained via a MAGMA computation [4], while we give here a character-theoretic proof.

*Proof.* First note that we may identify  $H$  with a subgroup of  $G$  and let  $P \in \text{Syl}_2(H)$ , so that  $P \in \text{Syl}_2(G)$  as well. Then  $P \cong D_8$  (the dihedral group of order 8),  $N := N_H(P) = N_G(P) \cong C_3 \times P$ , and  $N \leq H \leq G$ . In particular,  $X(N) \cong \mathbb{Z}/3\mathbb{Z}$ . By Lemma 3.3(b), we need to determine whether the  $kH$ - and  $kG$ -Green correspondents of the two non-trivial elements  $1a, 1a' \in X(N)$  are endo-trivial. So, set  $\mathfrak{f}_H := f_{(H, P, N)}$  and  $\mathfrak{f}_G := f_{(G, P, N)}$ . But  $1a' = 1a^*$ , and endo-triviality is preserved by passage to the  $k$ -dual, hence we only need to determine whether  $\mathfrak{f}_H^{-1}(1a)$  and  $\mathfrak{f}_G^{-1}(1a)$  are endo-trivial modules.

Note that the group  $3.\mathfrak{A}_6$  has a maximal subgroup  $M \cong C_3 \times \mathfrak{S}_4$  (indeed the product is direct because  $M(\mathfrak{S}_4) \cong C_2$ ) containing  $N$ , so that  $N \leq M \leq H \leq G$ . Hence

$$TT(M) = X(M) \cong \mathbb{Z}/3\mathbb{Z}$$

and we denote by  $1b$  and  $1b^*$  the two non-trivial elements of  $X(M)$ , which we identify with their ordinary characters.

(a) We may assume that  $\mathfrak{f}_H^{-1}(1a) \mid 1b \uparrow_M^H$ . Then we calculate that

$$\chi_{1b} \uparrow_M^H = \chi_{6_1} + \chi_{9_2}.$$

Thus  $\dim_k(\text{End}_{kH}(1b \uparrow_M^H)) = 2$  by making use of Scott's Theorem in [34] (see also [23, II Theorem 12.4(i) and I Lemma 14.5]). Then it follows from the 2-decomposition matrix of  $H$  [40,  $A_6(\text{mod } 2)$ ] that

$$1b \uparrow_M^H = 9_2 \oplus \begin{pmatrix} 3c \\ 3d \end{pmatrix}$$

as  $kH$ -modules, where  $3c$  and  $3d$  are the two non-isomorphic simple modules of dimension 3 belonging to the 2-block containing  $9_2$ , and  $\begin{pmatrix} 3c \\ 3d \end{pmatrix}$  is a uniserial  $kH$ -module with composition factors  $3c$  and  $3d$ . Therefore  $\mathfrak{f}_H^{-1}(1b) = 9_2$ , which affords  $\chi_{9_2} \in \text{Irr}(H)$ . Hence  $\mathfrak{f}_H^{-1}(1b)$  is endo-trivial by Theorem 3.4 as it takes value 1 on any non-trivial 2-element of  $H$ , see [14, p.5]. As a consequence,  $TT(H) \cong X(N) \cong \mathbb{Z}/3\mathbb{Z}$ .

(b) Now passing to  $G$ ,  $\mathfrak{f}_G^{-1}(1a) \mid f_M^{-1}(1b) \uparrow_H^G$ , where  $f_M := f_{(H, P, M)}$  (recall that, on the other hand,  $\mathfrak{f}_G$  denotes the Green correspondence with respect to  $(G, P, N)$ ). We calculate that

$$\chi_{9_2} \uparrow_H^G = \chi_{15_4} + \chi_{24_1} + \chi_{24_3}.$$

But  $\chi_{24_1}, \chi_{24_3} \in \text{Irr}(G)$  being defect-zero characters, we obtain that  $\mathfrak{f}_G^{-1}(1a)$  affords  $\chi_{15_4} \in \text{Irr}(G)$ , which does not take value 1 on the unique conjugacy class of involutions, see [14, p.10]. Thus, by Theorem 3.4,  $\mathfrak{f}_G^{-1}(1a)$  and  $\mathfrak{f}_G^{-1}(1a')$  are not endo-trivial modules, and we must have  $TT(G) = \{[k_G]\}$ .

(c) By [40,  $A_6.2_2 \pmod{2}$ ],  $R$  has a unique block of full defect, namely the principal block. Therefore it follows from Remark 3.1 that  $TT(R) = TT_0(R)$  and we conclude from Corollary 6.2 and the proof of Proposition 6.1 that  $TT(R) = X(R) = \{[k_R]\}$ .  $\square$

**Corollary 6.4** (See Remark 1.3.). *Set  $H := O_{2'}(G)$  and  $\overline{G} := G/H$ . Assume further that  $\overline{G} \cong \mathfrak{A}_6$  and  $G$  has a unique component  $E := E(G) \cong 3.\mathfrak{A}_6$ . Then, the following hold:*

- (a)  $G = HE$ ,  $[H, E] = 1$  (and hence  $G$  is a central product of  $H$  and  $E$ ), and moreover  $H \cap E = Z(E) \cong C_3$  and  $Z(G) \leq H = C_G(E)$ .
- (b) Let  $W$  be a simple  $kE$ -module such that  $[W] \in K(E)$  and  $\dim(W) = 9$  as in Lemma 6.3(a). Then, there exists a simple and trivial source  $kG$ -module  $V$  such that  $V \downarrow_E \cong W \oplus \cdots \oplus W$  ( $m$  summands) for a positive integer  $m$ . Further,  $V$  is endo-trivial if and only if  $W$  extends to  $V$ . In particular, if  $H^2(G/E, k^\times) = 1$ , then there exists a 9-dimensional simple  $kG$ -module whose class is in  $K(G)$ .

*Proof.* (a) Since  $E \triangleleft G$  and  $G/H$  is non-abelian simple, we have  $G = HE$ . Then,

$$\mathfrak{A}_6 \cong G/H = HE/H \cong E/(H \cap E) = (3\mathfrak{A}_6)/(H \cap E).$$

Hence  $C_3 \cong H \cap E = Z(E)$ . We have also  $Z(G) \leq H$  since  $Z(G) \triangleleft G$  and  $G/H$  is non-abelian simple. Further, if  $G = HC_G(E)$ , then  $[P, E] = 1$ , a contradiction, so that  $C_G(E) \leq H$  since  $C_G(E) \triangleleft G$  and  $G/H$  is non-abelian simple.

Next, we claim that  $H \leq C_G(E)$ . Take any  $h \in H$ , and let  $\phi_h$  be an element of  $\text{Aut}(E)$  given by  $y \mapsto h^{-1}yh$  for  $y \in E$ . Since  $|Z(E)| = 3$ ,  $\phi_h$  acts trivially on  $Z(E)$ , and hence we can consider that  $\phi_h \in \text{Aut}(E/Z(E)) = \text{Aut}(\mathfrak{A}_6)$ . Since  $|h|$  is odd and  $|\text{Out}(\mathfrak{A}_6)| = 4$  by [14, p.4], we know that  $\phi_h$  is an inner automorphism of  $\mathfrak{A}_6$ , and hence  $\phi_h$  is an inner automorphism of  $E$ . This implies that there is an element  $y_0 \in E$  with  $hy_0 \in C_G(E)$ . Since we know already that  $C_G(E) \leq H$ , we have  $y_0 \in H$ , so that  $y_0 \in H \cap E = Z(E) \leq C_G(E)$ . Therefore  $h \in C_G(E)$ .

(b) The first part follows easily from (a) and the Clifford Theorem. Then, the second part follows from Theorem 3.4. The final part follows immediately.  $\square$

## 7. TORSION ENDO-TRIVIAL MODULES IN THE DIHEDRAL CASE: PROOF OF THEOREM 1.2

We now turn to the general case and prove Theorem 1.2 of the introduction. Throughout this section we use the notations  $G$ ,  $P$  and  $\overline{G}$  as in §5, and further set  $H := O_{2'}(G)$ , and hence  $\overline{G} := G/H$  and  $\overline{P} := HP/H \cong P$ .

**Proposition 7.1.** *Let  $V$  be an indecomposable endo-trivial  $kG$ -module such that  $[V] \in K(G)$ . Then the following hold:*

- (a) *If  $\overline{G} \not\cong \mathfrak{A}_6$ , then  $\dim_k V = 1$ .*
- (b) *If  $\overline{G} \cong \mathfrak{A}_6$  and  $\dim_k V \neq 1$ , then  $\dim_k V = 9$ ,  $V$  is simple,  $[V^{\otimes 3}] \in X(G)$ .*

*Proof.* Since  $G$  has no strongly 2-embedded subgroups by Lemma 5.2, Theorem 4.4(a) yields that there exist a 2-cocycle  $\overline{\alpha} \in Z^2(\overline{G}, k^\times)$ , a one-dimensional  $k^\alpha G$ -module  $1b$  for  $\alpha := \text{Inf}_{\overline{G}}^G(\overline{\alpha})$  and a  $k^{\overline{\alpha}^{-1}}\overline{G}$ -module  $W$ , which we may regard as a  $k^{\alpha^{-1}}G$ -module via inflation from  $\overline{G}$  to  $G$ , such that

$$V \cong 1b \otimes_k W.$$

Now, we go through the possibilities for  $\overline{G}$  according to Hypotheses 5.1 and compute  $\dim_k(V)$  in each case.

To start with, if  $G$  satisfies one of the hypotheses (D1), (D5), (D6), (D7), or (D8), then  $H^2(\overline{G}, k^\times) = 1$  by Lemma 5.3. Therefore Theorem 1.1 yields

$$K(G) = X(G) + \text{Inf}_{\overline{G}}^G(K(\overline{G})).$$

Besides  $K(\overline{G}) = X(\overline{G})$  by Proposition 6.1. In consequence  $K(G) = X(G)$  and it follows that  $\dim_k(V) = 1$  in all cases.



Hence we may assume that  $G$  satisfies one of **(D2)**, **(D3)**, or **(D4)**. If  $[\bar{\alpha}]$  is trivial, then by the same argument as above we obtain  $\dim_k(V) = 1$ . Therefore we assume from now on that  $[\bar{\alpha}]$  is non-trivial and it follows from Lemma 5.3 that  $||[\bar{\alpha}]|| = 3$ . Then there exists a non-split central extension

$$(Z) : 1 \rightarrow \tilde{Z} \rightarrow \tilde{G} \rightarrow \bar{G} \rightarrow 1$$

where  $\tilde{Z} \cong C_3$ , and we write  $\tilde{G} = 3.\bar{G}$ , the triple cover of  $\bar{G}$ . Then it follows from Theorems of Schur [31, p.214, and Theorems 3.5.21 and 3.5.22] that the module  $W$  over  $k^{\bar{\alpha}^{-1}}\bar{G}$  corresponds to an indecomposable  $k\tilde{G}$ -module  $\widetilde{W}$  such that  $\widetilde{W} = W$  as  $k$ -vector spaces, and moreover, if  $\tilde{P} \in \text{Syl}_2(\tilde{G})$ , then  $W \downarrow_{k^{\bar{\alpha}^{-1}}\bar{P}} \cong \widetilde{W} \downarrow_{k\tilde{P}}$  as  $k\tilde{P}$ - (and also as  $k\bar{P}$ -) modules via the canonical isomorphism  $\tilde{P} \cong \bar{P} \cong P$ . We claim that  $[\widetilde{W}] \in K(\tilde{G})$ . Indeed, since  $[\alpha \downarrow_{H \rtimes P}] = 1$  by Theorem 4.4(a) and its proof, we have

$$V \downarrow_{H \rtimes P} = (1b) \downarrow_{k(H \rtimes P)} \otimes_k W \downarrow_{k(H \rtimes P)}$$

as a tensor product of (non-twisted)  $k(H \rtimes P)$ -modules. By Lemma 3.3(b),  $[V \downarrow_{H \rtimes P}] \in K(H \rtimes P)$  since  $[V] \in K(G)$  and  $\text{Res}_{H \rtimes P}^G$  is a group homomorphism. Therefore, since  $K(H \rtimes P)$  is a subgroup of  $T(H \rtimes P)$ , we have

$$[W \downarrow_{k(H \rtimes P)}] = [V \downarrow_{H \rtimes P}] - [(1b) \downarrow_{k(H \rtimes P)}] \in K(H \rtimes P) = TT(H \rtimes P),$$

where the latter equality of groups holds by Lemma 3.3(c). Thus, as  $H$  acts trivially on  $W$ , it follows from Lemma 3.2(c) that

$$[W \downarrow_{k((H \rtimes P)/H)}] \in TT((H \rtimes P)/H) = TT(\bar{P}) = K(\bar{P}),$$

where, again, the latter equality of groups holds by Lemma 3.3(c). But  $K(\bar{P}) = \{[k_{\bar{P}}]\}$  by definition since  $\bar{P}$  is a 2-group. Thus,  $[W \downarrow_{\bar{P}}] = [k_{\bar{P}}]$ , and hence  $[\widetilde{W} \downarrow_{\tilde{P}}] = [k_{\tilde{P}}]$ , so that by Lemma 3.2(a),  $\widetilde{W}$  itself is a trivial source endo-trivial  $k\tilde{G}$ -module, that is,  $[\widetilde{W}] \in K(\tilde{G})$ .

Now assume that  $G$  satisfies **(D4)**, i.e.  $\bar{G} \cong \text{PGL}(2, 9)$ . In this case,  $\tilde{G} = 3.\text{PGL}(2, 9)$ . Since  $\widetilde{W}$  is an indecomposable endo-trivial module, Lemma 6.3(c) yields  $\widetilde{W} = k_{\tilde{G}}$ . Thus

$$\dim_k V = \dim_k(1b \otimes_k W) = \dim_k W = \dim_k \widetilde{W} = 1.$$

Next assume that  $G$  satisfies **(D2)**, i.e.  $\bar{G} \cong \mathfrak{A}_7$ . In this case,  $\tilde{G} = 3.\mathfrak{A}_7$ . By the above, the  $k\tilde{G}$ -module  $\widetilde{W}$  is indecomposable endo-trivial such that  $[\widetilde{W}] \in K(\tilde{G})$ . Thus, Lemma 6.3(b) yields  $\widetilde{W} = k_{\tilde{G}}$ , so that  $\dim_k V = \dim_k W = \dim_k \widetilde{W} = 1$ .

Finally assume that  $G$  satisfies **(D3)**, i.e.  $\bar{G} \cong \mathfrak{A}_6$ , so that  $\tilde{G} = 3.\mathfrak{A}_6$ . Since  $\widetilde{W}$  is indecomposable endo-trivial and  $[\widetilde{W}] \in K(\tilde{G})$ , Lemma 6.3(a) implies that  $\widetilde{W} \in \{k_{\tilde{G}}, 9_1, 9_2\}$  (where  $9_1$  and  $9_2$  are as in Lemma 6.3(a)) and therefore is a simple module. Hence, again, we compute

$$\dim_k V = \dim_k(1b \otimes_k W) = \dim_k W = \dim_k \widetilde{W} \in \{1, 9\}.$$

Next, we claim that  $V$  is a simple module as well. We may assume, without loss of generality, that  $\widetilde{W} = 9_1$ , which lies in one of the two non-principal 2-blocks of full defect of  $3.\mathfrak{A}_7$ . So let  $B$  and  $\tilde{B}$  be the blocks of  $kG$  and  $k\tilde{G}$ , respectively, to which  $V$  and  $\widetilde{W}$  belong. Then,  $B$  and  $\tilde{B}$  correspond to each other by the Morita equivalence given by the result of Morita in [21, Lemma 2] (see also [30] and [31, Theorem 5.7.4]), and  $V$  and  $\widetilde{W}$  correspond to each other via this Morita equivalence. Hence  $V$  is a simple  $kG$ -module since  $\widetilde{W}$  is simple.

Now, by Theorem 4.4(b), we have that  $V^{\otimes 3} = (1b)^{\otimes 3} \otimes_k W^{\otimes 3}$  is the tensor product of a (non-twisted)  $kG$ -module  $(1b)^{\otimes 3}$  and a (non-twisted)  $kG$ -module  $W^{\otimes 3}$  such that  $[W^{\otimes 3}] \in \text{Inf}_{\overline{G}}^G(K(\overline{G}))$ . Since  $\overline{G} = \mathfrak{A}_6$ , we have  $K(\overline{G}) = \{[k_{\overline{G}}]\}$  by Proposition 6.1. Therefore  $[W^{\otimes 3}] = [k_G]$ , and hence  $[V^{\otimes 3}] = [(1b)^{\otimes 3}] \in X(G)$ .  $\square$

*Proof of Theorem 1.2.* Since the Sylow 2-subgroups of  $G$  are dihedral of order at least 8, Lemma 3.3(c) yields  $TT(G) = K(G)$ . Thus the claims follow from Proposition 7.1.  $\square$

**Example 7.2.** We now give an example of a group  $G$  with a dihedral Sylow 2-subgroup  $P$  of order 8, for which there exist classes  $[V] \in TT(G)$  such that

$$[V^{\otimes 3}] \in X(G) \setminus \{[k_G]\}.$$

We put ourselves in the situation of Corollary 6.4 and consider the following example. We define  $G$  to be the central product defined by

$$G := C_9 * 3.\mathfrak{A}_6 \cong (C_9 \times 3.\mathfrak{A}_6)/C_3,$$

so that  $\overline{G} := G/O_{2'}(G) \cong \mathfrak{A}_6$ , but  $H^2(\overline{G}, k^\times) \cong C_3$ . It easily follows that  $X(G) \cong \mathbb{Z}/3\mathbb{Z}$  and  $X(N_G(P)) \cong \mathbb{Z}/9\mathbb{Z}$ . Then, using GAP [5], we compute the following. First, inducing the nine linear characters of  $N_G(P)$  to  $G$ , we find that the  $kG$ -Green correspondents of the associated  $kN_G(P)$ -modules afford the three linear characters corresponding to  $X(G) \cong \mathbb{Z}/3\mathbb{Z}$  (this part is obvious) and six 9-dimensional ordinary characters  $\chi_{9_1}, \dots, \chi_{9_6}$ , reducing modulo 2 to 9-dimensional simple  $kG$ -modules  $9_1, \dots, 9_6$ , respectively (all lying in pairwise distinct blocks). Then Theorem 3.4 ensures that these modules are all endo-trivial since their ordinary characters take value one on any non-trivial 2-element of  $G$ . Hence we conclude that

$$TT(G) = K(G) \cong X(N_G(P)) \cong \mathbb{Z}/9\mathbb{Z}.$$

Finally we see that the characters  $\chi_{\widehat{9}_i} \otimes_K \chi_{\widehat{9}_i} \otimes_K \chi_{\widehat{9}_i}$  for each  $1 \leq i \leq 6$  have no trivial constituents, so that  $[(9_i)^{\otimes 3}] \in X(G) \setminus \{[k_G]\}$  for each  $1 \leq i \leq 6$ .

## 8. GROUPS WITH KLEIN-FOUR SYLOW 2-SUBGROUPS REVISITED

The purpose of this section is to prove Theorem 1.5. Throughout this section we assume that  $P$  is a Klein-four Sylow 2-subgroup of  $G$ , that is  $P \cong C_2 \times C_2$ , and let  $N := N_G(P)$ . Furthermore, we use the notations  $H := O_{2'}(G)$ ,  $\overline{G} := G/H$  and  $\overline{P} := (HP)/H = (H \rtimes P)/H \cong P$ .

**Lemma 8.1.** *One of the following holds:*

- (1)  $\overline{G} \cong P$ .
- (2)  $\overline{G} \cong \mathfrak{A}_4 \cong \text{PSL}(2, 3)$ ,
- (3)  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q = r^m$  is a power of an odd prime  $r$  with  $3 < q \equiv \pm 3 \pmod{8}$ ,  $f$  is an odd integer, and  $C_f \leq \text{Gal}(\mathbb{F}_q/\mathbb{F}_r) \cong C_m$ .

*Proof.* This follows from [39, Theorem I], see also [36, Proof of Theorem 6.8.7, (8.9) in Chapter 6 and Theorem 6.8.11].  $\square$

**Hypotheses 8.2.** For the purposes of our computations, we split case (3) above in further subcases and say that  $G$  satisfies the hypothesis:

- (K1) if  $\overline{G} \cong P$ ;
- (K2) if  $\overline{G} \cong \mathfrak{A}_4 \cong \text{PSL}(2, 3)$ ;

- (K3) if  $\overline{G} \cong \mathfrak{A}_5 \cong \text{PSL}(2, 5)$ ;
- (K4) if  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q = r^m$  is a power of an odd prime  $r$  with  $3 < q \equiv 3 \pmod{8}$  and  $f$  is odd with  $f|m$  (that is  $C_f \leq \text{Gal}(\mathbb{F}_q/\mathbb{F}_r) \cong C_m$ );
- (K5) if  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q = r^m$  is a power of an odd prime  $r$  with  $5 < q \equiv 5 \pmod{8}$  for an odd  $f$  with  $f|m$  (that is  $C_f \leq \text{Gal}(\mathbb{F}_q/\mathbb{F}_r) \cong C_m$ ).

**Lemma 8.3.** *The following two conditions are equivalent:*

- (1)  $G$  has a strongly 2-embedded subgroup.
- (2)  $\overline{G} \cong \mathfrak{A}_5$ .

Moreover, if (1) holds, then any strongly 2-embedded subgroup  $G_0$  of  $G$  is of the form  $HN$  where  $N := N_G(P)$  for a suitable choice of  $P \in \text{Syl}_2(G)$ , so that  $G_0/O_{2'}(G_0) \cong \mathfrak{A}_4$ .

*Proof.* This follows from the Bender-Suzuki Theorem [36, Theorem 6.4.2(2)].  $\square$

**Lemma 8.4.** *The group  $H^2(\overline{G}, k^\times)$  is trivial.*

*Proof.* Set  $h := |H^2(\overline{G}, k^\times)|$ . It follows from [41, Proposition 3.2] (see also [21, Lemma 5] and [31, Lemma 3.5.4(ii)]) that  $|H^2(\overline{G}, k^\times)| = |M(\overline{G})|_{2'}$  since  $k$  has characteristic 2.

If (K1) holds, then  $h = 1$  by [31, Theorem 2.7.4]. Otherwise, by Lemma 8.1, we have  $\overline{G} \cong \text{PSL}(2, q) \rtimes C_f$ , where  $q$  is a power of an odd prime with  $q \equiv \pm 3 \pmod{8}$ , and  $f$  is odd. First, note that  $|M(\text{PSL}(2, q))| = 2$ , see [20, Theorem 4.9.1(ii)]. If  $q = 3$ , then  $\overline{G} \cong \text{PSL}(2, 3) \cong \mathfrak{A}_4$  and  $f = 1$ , so that the assertion holds.

So we may assume  $q > 3$ , and hence  $\text{PSL}(2, q)$  is non-abelian simple as is well known. Since  $\overline{G}/\text{PSL}(2, q)$  is cyclic and  $\text{PSL}(2, q)$  is perfect, it follows from [19, Theorem 3.1(i)] that  $|M(\overline{G})| \mid |M(\text{PSL}(2, q))| = 2$ , so that  $h = 1$ .  $\square$

**Lemma 8.5.** *Suppose that  $\theta \in \text{Irr}(H)$  and that  $V$  is an indecomposable  $kG$ -module such that  $V \downarrow_H$  contains  $\theta$  as a constituent. If  $\overline{G} \not\cong \mathfrak{A}_5$  and  $V$  is endo-trivial, then  $\theta$  is  $G$ -invariant.*

*Proof.* If  $\overline{G} \not\cong \mathfrak{A}_5$ , then by Lemma 8.3,  $G$  has no strongly 2-embedded subgroups. Therefore the claim follows from Lemma 4.3.  $\square$

**Proposition 8.6.** *If  $H = 1$ , then one of the following five cases holds:*

- (a) If (K1) holds, then  $K(G) = \{[k_G]\}$ .
- (b) If (K2) holds, then  $K(G) = X(G) = \{[k], [1_\omega], [1_{\omega^2}]\} \cong \mathbb{Z}/3\mathbb{Z}$ , where  $1_\omega$  and  $1_{\omega^2}$  are the two non-trivial one-dimensional  $kG$ -modules.
- (c) If (K3) holds, then  $K(G) = \{[k], [5i], [(5i)^*]\} \cong K(\mathfrak{A}_4) \cong \mathbb{Z}/3\mathbb{Z}$ , and where  $5i, (5i)^*$  are uniserial, trivial source, and endo-trivial  $kG$ -modules in  $B_0(G)$ , both affording the unique irreducible character of degree 5,  $\chi_5 \in \text{Irr}(G)$ . (See [22, Lemma 4.1].)
- (d) If (K4) holds, then  $K(G) \cong X(G) \oplus TT_0(G)$ , where  $X(G) \cong \mathbb{Z}/f\mathbb{Z}$  and  $TT_0(G) = \langle [(q-1)/2] \rangle \cong \mathbb{Z}/3\mathbb{Z}$  for  $(q-1)/2$  a simple trivial source endo-trivial  $kG$ -module of dimension  $(q-1)/2$  affording an irreducible character  $\chi_{(q-1)/2} \in \text{Irr}(B_0(G))$ .
- (e) If (K5) holds, then  $K(G) \cong X(G) \oplus TT_0(G)$ , where  $X(G) \cong \mathbb{Z}/f\mathbb{Z}$  and  $TT_0(G) = \langle [V] \rangle \cong \mathbb{Z}/3\mathbb{Z}$  for a trivial source endo-trivial  $kG$ -module  $V$  such that  $V$  is uniserial of length 3 with composition factors  $((q-1)/2)a, k_G, ((q-1)/2)b$ , where  $((q-1)/2)a$  and  $((q-1)/2)b$  are non-isomorphic simple  $kG$ -modules in  $B_0(G)$  of dimension  $(q-1)/2$ , and  $V$  affords the Steinberg character  $\text{St}_G \in \text{Irr}(B_0(G))$  of degree  $q$ .

*Proof.* (a) is clear since  $G \cong C_2 \times C_2$  is a 2-group. Next  $K(\mathfrak{A}_4) \cong K(\mathfrak{A}_5) \cong \mathbb{Z}/3\mathbb{Z}$  by [8, Theorem 4.2 and its proof]. Moreover  $K(\mathfrak{A}_4) \cong X(\mathfrak{A}_4)$  by Lemma 3.3(a) because  $C_2 \times C_2$  is normal in  $\mathfrak{A}_4$ , and the structure of the modules in  $K(\mathfrak{A}_5)$  is given by [22, Lemma 4.1]. Hence (b) and (c) hold.

If **(K4)** or **(K5)** holds, then  $K(G) \cong \mathbb{Z}/f\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  with  $X(G) \cong \mathbb{Z}/f\mathbb{Z}$  and  $TT_0(G) \cong \mathbb{Z}/3\mathbb{Z}$  by [22, Theorem 1.4(d) and its proof]. Finally the structure of a generator  $[V]$  of  $TT_0(G)$  with  $V$  indecomposable is obtained as follows. By [22, Proposition 4.3(a)], the lift of  $V$  to characteristic zero affords an irreducible character  $\chi_V \in \text{Irr}(B_0(G))$ , so that using Theorem 3.4, by investigation of the generic character table of  $\text{PSL}(2, q)$  (see e.g. [3, Table 5.4]) we see that  $\chi_V(1) = (q-1)/2$  if **(K4)** holds, and  $\chi_V(1) = q$  if **(K5)** holds. Then the composition factors of  $V$  follow from the 2-decomposition matrix of  $\text{PSL}(2, q)$ , see [3, Table 9.1].  $\square$

*Proof of Theorem 1.5.* First  $TT_0(G) \cong \mathbb{Z}/3\mathbb{Z}$  by [22, Theorem 1.4 and its proof], and any indecomposable torsion endo-trivial module lifts to an irreducible ordinary character by [22, Theorem 1.1(a)]. In addition, since  $P \cong C_2 \times C_2$ ,  $TT(G) = K(G)$  by Lemma 3.3(c).

Assume that **(K1)** holds, then  $G$  is 2-nilpotent, and [32, Theorem] yields  $K(G) = X(G)$ , as was conjectured in [9, Conjecture 3.6]. Hence (a) holds.

Assume that **(K2)** holds, then  $G$  is solvable by the Feit-Thompson Theorem, so that  $K(G) = X(G)$  by [9, Theorem 6.2(2)]. Hence (b) holds.

Assume that **(K3)** holds. Then, by Lemma 8.3,  $G$  has a strongly 2-embedded subgroup  $G_0$  such that  $G_0/O_{2'}(G_0) \cong \mathfrak{A}_4$ . Therefore  $K(G) \cong K(G_0)$  by Lemma 3.5 and  $K(G_0) \cong X(G_0)$  by (b). The non-trivial indecomposable torsion endo-trivial  $kG$ -modules in  $B_0(G)$  have dimension 5 by Proposition 8.6(c). Hence (c) holds.

Assume that **(K4)** or **(K5)** holds. Then  $G$  has no strongly 2-embedded subgroups, and  $H^2(G, k^\times) = 1$  by Lemmas 8.3 and 8.4. Therefore, by Theorem 1.1,

$$K(G) = X(G) + K(\overline{G}),$$

where we identify  $K(\overline{G})$  with  $\text{Inf}_{\overline{G}}^G(K(\overline{G}))$ . In addition, by Proposition 8.6(d) and (e), we have  $K(\overline{G}) = X(\overline{G}) \oplus TT_0(\overline{G}) \cong \mathbb{Z}/f\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ . Therefore

$$K(G) = X(G) + X(\overline{G}) + TT_0(\overline{G}) = X(G) + TT_0(G)$$

since  $\text{Inf}_{\overline{G}}^G(TT_0(\overline{G})) = TT_0(G)$  and  $\text{Inf}_{\overline{G}}^G(X(\overline{G})) \leq X(G)$ . But  $X(G) \cap TT_0(G) = \{[k_G]\}$  by Proposition 8.6(d) and (e), thus

$$K(G) \cong X(G) \oplus TT_0(G) \cong X(G) \oplus \mathbb{Z}/3\mathbb{Z}.$$

Hence (d) holds.  $\square$

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## REFERENCES

- [1] M. AUSLANDER, J.F. CARLSON, Almost-split sequences and group rings. *J. Algebra* **103** (1986), 122–140.
- [2] H. BENDER, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt. *J. Algebra* **17** (1971), 527–554.
- [3] C. BONNAFÉ, Representations of  $SL_2(\mathbb{F}_q)$ . Springer-Verlag London, Ltd., London, 2011.
- [4] W. BOSMA, J. CANNON AND C. PLAYOUST, The Magma algebra system I: the user language'. *J. Symbolic Comput.* **24** (1997), 235–265.
- [5] T. BREUER, The GAP Character Table Library, Version 1.2.2. GAP package, <http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib>.
- [6] J.F. CARLSON, Endotrivial modules. *Proc. Symp. Pure Math.* **86** (2012), 99–111.
- [7] J.F. CARLSON, N. MAZZA, D. NAKANO, Endotrivial modules for finite groups of Lie type. *J. reine angew. Math.* **595** (2006), 93–119.
- [8] J.F. CARLSON, N. MAZZA, D. NAKANO, Endotrivial modules for the symmetric and the alternating groups. *J. Edinburgh. Math. Soc.* **52** (2009), 45–66.
- [9] J.F. CARLSON, N. MAZZA, J. THÉVENAZ, Endotrivial modules for  $p$ -solvable groups. *Trans. Amer. Math. Soc.* **363** (2011), 4979–4996.
- [10] J.F. CARLSON, N. MAZZA, J. THÉVENAZ, Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup. *J. Eur. Math. Soc.* **15** (2013), 157–177.
- [11] J.F. CARLSON, N. MAZZA, J. THÉVENAZ, Torsion-free endotrivial modules. *J. Algebra* **398** (2014), 413–433.
- [12] J.F. CARLSON, J. THÉVENAZ, Torsion endo-trivial modules. *Algebr. Represent. Theory* **3** (2000), 303–335.
- [13] J.F. CARLSON, J. THÉVENAZ, The torsion group of endotrivial modules. *Algebra Number Theory* **9** (2015), 749–765.
- [14] J.H. CONWAY, R.T. CURTIS, S.P. NORTON, R.A. PARKER, R.A. WILSON, Atlas of Finite Groups. Clarendon Press, Oxford, 1985.
- [15] E.C. DADE, Endo-permutation modules over  $p$ -groups I and II. *Ann. of Math.* **108** (1978), 317–346.
- [16] D. GORENSTEIN, Finite Group Theory. Harper and Row, New York, 1968.
- [17] D. GORENSTEIN, The Classification of Finite Simple Groups Vol.1: Groups of Noncharacteristic 2 type. Plenum Press, New York and London, 1983.
- [18] D. GORENSTEIN, J.H. WALTER, The characterization of finite groups with dihedral Sylow 2-subgroups. I, II, III. *J. Algebra* **2** (1965), 85–151, 218–270, 354–393.
- [19] M.R. JONES, Some inequalities for the multiplier of a finite group II. *Proc. Amer. Math. Soc.* **45** (1974), 167–172.
- [20] G. KARPILOVSKI. Projective Representations of Finite Groups. Dekker, New York, 1985.
- [21] S. KOSHITANI, A remark on blocks with dihedral defect groups in solvable groups. *Math. Z.* **179** (1982), 401–406.
- [22] S. KOSHITANI, C. LASSUEUR, Endo-trivial modules for finite groups with Klein-four Sylow 2-subgroups. To appear in *Manuscripta Math.* DOI: 10.1007/s00229-015-0739-5.
- [23] P. LANDROCK, Finite Group Algebras and their Modules. London Math. Soc. Lecture Notes **vol.84**, Cambridge University Press, Cambridge.
- [24] C. LASSUEUR, G. MALLE, Simple endotrivial modules for the linear, unitary and exceptional groups. To appear in *Math. Z.* (2015), DOI:10.1007/s00209-015-1465-0.
- [25] C. LASSUEUR, G. MALLE, E. SCHULTE, Simple endotrivial modules for quasi-simple groups. To appear in *J. reine angew. Math.* (2013), DOI:10.1515/crelle-2013-0100.
- [26] C. LASSUEUR, N. MAZZA, Endotrivial modules for the sporadic simple groups and their covers. *J. Pure Appl. Algebra* **219** (2015), 4203–4228.
- [27] C. LASSUEUR, N. MAZZA, Endotrivial modules for the Schur covers of the symmetric and alternating groups. To appear in *Algebr. Represent. Theory* (2015), DOI:10.1007/s10468-015-9542-y.
- [28] A. MARCUS, Representation Theory of Group Graded Algebras. Science Publishers, Inc., New York, 1999.
- [29] N. MAZZA, J. THÉVENAZ, Endotrivial modules in the cyclic case. *Arch. Math.* **89** (2007), 497–503.
- [30] K. MORITA, On group rings over a modular field which possess radicals expressible as principal ideals. *Sci. Rep. Tokyo Bunrika Daigaku* **A4** (1951), 177–194.



- [31] H. NAGAO, Y. TSUSHIMA, Representations of Finite Groups. Academic Press, New York, 1989.
- [32] G. NAVARRO, G.R. ROBINSON, On endo-trivial modules for  $p$ -solvable groups. Math. Z. **270** (2012), 983–987.
- [33] G.R. ROBINSON, On simple endotrivial modules. Bull. London Math. Soc. **43** (2011), 712–716.
- [34] L.L. SCOTT, Modular permutation representations. Trans. Amer. Math. Soc. **175** (1973), 101–121.
- [35] M. SUZUKI, On a class of doubly transitive groups II. Ann. of Math. (2) **79** (1964), 514–589.
- [36] M. SUZUKI, Group Theory II. Springer, Heidelberg, 1986.
- [37] J. THÉVENAZ,  $G$ -Algebras and Modular Representation Theory. Clarendon Press, Oxford, 1995.
- [38] J. THÉVENAZ, Endo-permutation modules, a guided tour, in: Group Representation Theory. EPFL Press, Lausanne, 2007, pp.115–146.
- [39] J.H. WALTER, The characterization of finite groups with abelian Sylow 2-subgroups. Ann. of Math. **89** (1969), 405–514.
- [40] R. WILSON, J. THACKRAY, R. PARKER, F. NOESKE, J. MÜLLER, F. LÜBECK, C. JANSEN, G. HISS, T. BREUER, The Modular Atlas Project. <http://www.math.rwth-aachen.de/~MOC>.
- [41] K. YAMAZAKI, On projective representations and ring extensions of finite groups. J. Fac. Sci. Univ. Tokyo **10** (1964), 147–195.

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